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MATHEMATICAL ANALYSIS OF NAVIER-STOKES EQUATIONS IN FLUID DYNAMICS

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Abstract

The Navier-Stokes equations are the foundation of fluid dynamics because they provide a complete framework for describing the motion of viscous fluid substances since they are the cornerstone of the field. An examination of these equations from a mathematical perspective is presented in this work, with a particular emphasis on the fundamental characteristics, solution approaches, and applications of these equations. Beginning with the conservation rules of mass, momentum, and energy, we will investigate how the equations were derived from those specific laws. In the framework of the Millennium Prize Problem, important mathematical elements such as existence, uniqueness, and regularity of solutions are investigated. This serves to bring to light many unanswered topics and substantial obstacles that are now being faced in the area. In addition, we look into sophisticated numerical approaches for approximating solutions, such as finite difference, finite element, and spectral methods. These methods have applications in a wide variety of fields, including aerodynamics and weather modelling. In addition to this, the study delves into the dynamic relationship between analytical and computational methods when it comes to addressing turbulence, which is one of the most complicated phenomena that is governed by the Navier-Stokes equations. Through our study, we have demonstrated that these equations play a crucial part in bridging the gap between theoretical understanding and practical applications in the field of fluid dynamics. This offers potential avenues for further research in the fields of mathematical physics and engineering.

Keywords: *Mathematical, Navier-Stokes, Dynamics*

Introduction

Applications of fluid dynamics may be found in a wide variety of fields, including engineering, meteorology, oceanography, and several others. Fluid dynamics plays an important part in understanding and forecasting the behaviour of fluids that are in motion. The Navier-Stokes equations are the foundation of this field of study. These equations are used to explain the motion of viscous fluid substances by means of a set of nonlinear partial differential equations that are derived from the fundamental concepts of energy conservation, momentum conservation, and mass conservation. These equations were initially developed in the 19th century by Claude-Louis Navier and George Gabriel Stokes. Since then, they have developed into an essential component of both theoretical and applied fluid mechanics. Even though they are used in a broad variety of contexts, the Navier-Stokes equations continue to be the topic of extensive research due to their complex mathematical nature. Questions about the existence of solutions and the smoothness of solutions for generic initial and boundary conditions, which are notably defined as one of the Millennium Prize Problems, highlight the relevance and complexity of this subject matter. In order to present a full mathematical examination of the Navier-Stokes equations, the purpose of this work is to highlight their derivation, basic features, and the difficulties that are connected with finding solutions. Particular focus is placed on the dynamic relationship that exists between analytical approaches and numerical techniques, which, when used together, contribute to the advancement of our understanding of hydraulic motion. Furthermore, we discuss the phenomena of turbulence, which is a defining characteristic of fluid dynamics that is regulated by these equations, as well as the consequences that this phenomenon has for both theoretical research and practical applications. The purpose of this paper is to make a contribution to the continuing discussion on the Navier-Stokes equations by bridging the gap between mathematical rigour and real-world applicability. In doing so, the study hopes to give insights into the deep implications that these equations have for the fields of science and engineering.

Derivation of the Navier-Stokes Equations and Preliminary Considerations

When modelling a fluid, F, using continuum mechanics, it is assumed that, over the specified time period, I ≡ [0, T], where T is greater than zero, F continually fills an area, **Ω**, of the three-dimensional space, R3. For the material points (or particles), material surfaces, and material volumes, respectively, we refer to them as material points, material surfaces, and material volumes. Within the majority of applications that are pertinent, the area **Ω** does not depend on time. This occurs in particular if the fluid is confined by rigid walls, such as, for example, when a flow is passing through a rigid obstruction or when a flow is occurring in a constrained container with fixed walls. On the other hand, there are also certain noteworthy instances in which the value of **Ω** is associated with the passage of time. One such instance is the movement of a fluid within a pipe that has elastic walls. Over the course of this chapter, we will be discussing flow of F in which **Ω** is not reliant on the passage of time.

Keep the Laws in Balance. It is more convenient to represent the essential physical quantities in the Eulerian form in order to explain the motion of F rather than using the Eulerian form. To be more specific, if x = (x1, x2, x3) is a point in **Ω** and t is a point in the interval [0, T], then we will let ρ = ρ(x, t), v = v(x, t)=(v1(x, t), v2(x, t), v3(x, t), and a = a(x, t)=(a1(x, t), a2(x, t), a3(x, t)) represent the density, velocity, and acceleration, respectively, of the particle of F that passes through the point x at the time t. As an additional point of interest, we indicate the external force per unit volume (body force) that is acting on F by the equation f = f(x, t)=(f1(x, t), f2(x, t), f3(x, t).

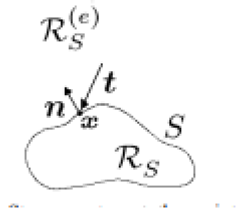
Within the framework of the continuum theory of non-polar fluids, it is postulated that the following equations must be valid in any circumstance of motion.

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t}(\boldsymbol{x},t)+\frac{\partial}{\partial x_i}(\rho(\boldsymbol{x},t)v_i(\boldsymbol{x},t))=0\,,\\ &\rho(\boldsymbol{x},t)\,a_i(\boldsymbol{x},t)=\frac{\partial T_{ji}}{\partial x_j}+\rho(\boldsymbol{x},t)\,f_i(\boldsymbol{x},t)\,,\quad i=1,2,3\\ &T_{ij}(\boldsymbol{x},t)=T_{ji}(\boldsymbol{x},t)\,,\quad i,j=1,2,3\,,\end{aligned}\right\}\ \text{for all } (\boldsymbol{x},t)\in\Omega\times(0,T)\,. \qquad (1)$$

The balancing rules of F in the Eulerian description are represented by these equations, which are in the local form. To be more specific, (1)1 is an expression of the principle of mass conservation, (1)2 is a provision of the balance of linear momentum, and (1)3 is comparable to the balance of angular momentum. Furthermore, the function T = T (x, t) = Tji(x, t) is a symmetric tensor field of the second order, known as the Cauchy stress tensor. This tensor field takes into consideration the internal forces that are exerted by the fluid system. To be more specific, let S represent a fixed, closed surface in R3 that is sufficiently smooth and that bounds the area RS. Let x be any point on S, and let n = n(x) be the outward unit normal to S at x. Moreover, let R(e) S be the area that is external to RS, where R(e) S is a subset of Ω 6 and is equal to ∅. Following this, the vector t = t(x, t) is defined as

$$\boldsymbol{t}(\boldsymbol{x},t) := \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{T}(\boldsymbol{x},t)\,, \qquad (2)$$

indicates the force that is applied on S by the fraction of the fluid that is contained inside R(e) S at the place x and at the time t; see Figure 1. There is a constitutive equation. The stretching tensor field, denoted by the equation D = D(x, t), is an essential kinematical parameter that is connected with the motion of F. The components of this field, denoted by Dij, are defined as follows:



See Figure 1 for an illustration of the stress vector at the surface S point x..

$$D_{ij} = \tfrac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\,, \quad i,j = 1,2,3\,. \qquad (3)$$

It should come as no surprise that the stretching tensor is symmetric, and in a general sense, it captures the rate at which regions of F undergo deformation. In point of fact, it is possible to demonstrate that a necessary and sufficient condition for a motion of F to be rigid is that the mutual distance between two arbitrary particles of F does not vary throughout the course of time. is that D(x, t)=0 at each (x, t) ∈ Ω×I. Moreover, divv(x, t) ≡ trace $\boldsymbol{D}(\boldsymbol{x},t) \equiv \frac{\partial v_i}{\partial x_i}(\boldsymbol{x},t) = 0$ for all $(\boldsymbol{x},t) \in \Omega \times I$, only in the event that the motion is isochoric, which means that the volume of every substance does not vary throughout the course of time. Among the many classes of fluids, the class of fluids with constant density is particularly remarkable since their general motion is isochoric. In point of fact, if ρ is a positive constant, then we may deduce from (1)1 that the value of divv(x, t) is equal to zero for all (x, t) that belong to the set Ω×I. The term "incompressible" refers to fluids that have a constant density. These incompressible fluids will be referred to as liquids throughout the course of this chapter.

It is generally the case that the internal forces will cause a deformation of the components of F while the generic motion is taking place. The term "constitutive equation" refers to the relationship that exists between internal forces and deformation, more specifically the functional connection that exists between T and D. This equation is responsible for defining the physical characteristics of the fluid. A liquid is said to be Newtonian if and only if the relationship between temperature and density is linear. This means that there is a scalar function p = p(x, t) (which represents the pressure) and a constant μ (which represents the shear viscosity) that exists in such a way that the pressure and the shear viscosity are linearly related.

$$T = -pI + 2\mu D \,, \tag{4}$$

I represents the identity matrix in this context. One makes the assumption that the shear viscosity of a viscous Newtonian liquid fulfils the limitation under consideration.

$$\mu > 0 \,. \tag{5}$$

When we return to the topic of the significance of this assumption, we shall do so in Remark 2. Navier-Stokes Equations are used. Given the fact that the condition div v = 0 is true, it is not difficult to deduce from (4) that

$$\frac{\partial T_{ji}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \Delta v_i$$

where $\Delta := \frac{\partial^2}{\partial x_l \partial x_l}$ refers to the operator of Laplace. Therefore, by using (1) and (4), we are able to draw the conclusion that the equations that control the motion of a Newtonian viscous liquid are provided by

$$\left.\begin{aligned} &\rho\, a_i = -\frac{\partial p}{\partial x_i} + \mu \Delta v_i + \rho\, f_i \,, \quad i = 1,2,3 \\ &\frac{\partial v_i}{\partial x_i} = 0 \end{aligned}\right\} \text{ in } \Omega \times (0,T) \,. \tag{6}$$

The fact that both equations in equation (6) are linear in all kinematical variables is a noteworthy observation to have in mind. Nevertheless, according to the Euler description, the acceleration is a nonlinear function of the velocity, and we get to the conclusion that

$$a_i = \frac{\partial v_i}{\partial t} + v_l \frac{\partial v_i}{\partial x_l} \,,$$

or, in a vector form,

$$a = \frac{\partial v}{\partial t} + v \cdot \nabla v \,.$$

Obtaining the Navier-Stokes equations by substituting this latter expression in equation (6) is as follows:

$$\left.\begin{aligned} &\rho \left(\frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p + \mu \Delta v + \rho f \,, \\ &\operatorname{div} v = 0 \end{aligned}\right\} \text{ in } \Omega \times (0,T) \,. \tag{7}$$

The provided parameters in these equations are the shear viscosity μ (which satisfies equation (5)), the force f, and the density ρ, which is constant. The unknowns are the velocity v = v(x, t) and the pressure p = p(x, t) fields. It would be appropriate to make some preliminary remarks on the equations used above. In point of fact, it is important to take note of the fact that the unknowns v and p do not occur in a symmetrical manner. To put it another way, the equation that describes the conservation of mass, which is (7)2, does not include the pressure field. Due to the fact that, from a mechanical standpoint, the pressure acts as a response force (Lagrange multiplier) coupled with the isochoricity restriction divv = 0, this is the reason why this is the case. To put it another way, anytime a section of the liquid decides to change its volume, the liquid responds by reacting with a pressure distribution that is appropriate in order to maintain the volume that was previously determined. In light of this, it is necessary to typically infer the pressure field in terms of the velocity field, after the latter has been established; for further information, go to Remark 1. Initial-Boundary Value Problem (1). In order to identify solutions to the issue (7), we need to add proper beginning circumstances, initial conditions at time t = 0, and boundary conditions. In point of fact, these requirements could be different depending on the particular physical situation that we are making an effort to simulate. Assuming that the region of flow, denoted by **Ω**, is surrounded by stiff walls, denoted by ∂**Ω**, and that the liquid does not slide at **Ω**, we will proceed with our analysis. When this occurs, the beginning and boundary conditions that are appropriate become, respectively, the ones that are listed below.

$$\begin{aligned} &\boldsymbol{v}(\boldsymbol{x},0)=\boldsymbol{v}_0(\boldsymbol{x})\,, \quad \boldsymbol{x}\in\Omega \\ &\boldsymbol{v}(\boldsymbol{x},t)=\boldsymbol{v}_1(\boldsymbol{x},t)\,, \quad (\boldsymbol{x},t)\in\partial\Omega\times(0,T)\,, \end{aligned} \tag{8}$$

in which v0 and v1 are vector fields that have been specified. Boundary-value problem and steady-state flow are both discussed. A significant and unique category of solutions to equation (7), which are referred to as steady-state solutions, is the case in which the velocity and pressure fields are not dependent on the passage of time. Naturally, one of the prerequisites that must be met in order for such solutions to be considered viable is that f must not be dependent on time. Consequently, we may deduce from equation (7) that a general steady-state solution (v = v(x), p = p(x)), must satisfy the following equations

$$\left.\begin{aligned} &\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v}=-\nabla p+\mu\Delta\boldsymbol{v}+\rho\boldsymbol{f}\,, \\ &\operatorname{div}\boldsymbol{v}=0 \end{aligned}\right\}\ \text{in}\ \Omega\,. \tag{9}$$

In light of the assumptions that have been made regarding the flow area, it can be shown from equation (8)2 that the boundary conditions that are suitable are

$$\boldsymbol{v}(\boldsymbol{x})=\boldsymbol{v}_*(\boldsymbol{x})\,, \quad \boldsymbol{x}\in\partial\Omega\,, \tag{10}$$

where v∗ is a prescribed vector field.

There is a flow in two dimensions. Separate attention deserves two-dimensional solutions that describe the planar movements of F in a number of mathematical concerns that are connected to the unique solvability of problems (7), (8), and (9) and ten. For these solutions, the variables v and p are dependent solely on x1, x2 (for example), and t in the case of (7) and (8). Furthermore, v3 is greater than or equal to 0. Therefore, the relevant (spatial) region of motion, denoted by the symbol **Ω**, is transformed into a subset of R2.

The first remark We discover that, for any time t (0, T), the pressure field p = p(x, t) must fulfil the following Neumann problem. This is determined by explicitly acting with div " on both sides of (7)1 and by taking into consideration (8)2.

$$\Delta p = \rho(\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \boldsymbol{f}) \quad \text{in } \Omega$$
$$\frac{\partial p}{\partial n} = -[\mu \Delta \boldsymbol{v} - \rho(\boldsymbol{v}_1 \cdot \nabla \boldsymbol{v} - \boldsymbol{f})] \cdot \boldsymbol{n} \quad \text{at } \partial\Omega \tag{11}$$

The conclusion that can be drawn from this is that the prescription of the pressure at the enclosing walls or at the beginning time independently of v could be incompatible with (8), and as a result, it might make the issue ill-posed. The second remark It is possible for us to provide a straightforward qualitative explanation of the assumption (5). Let us assume a steady-state flow of a viscous liquid between two parallel, stiff barriers Π1, Π2, which are positioned at a distance d apart and parallel to the x1 − x2 plane. If we look at Figure 2, we can see that this flow is characterised by a steady-state flow. In order to produce flow, a force per unit area F = Fe1, F > 0, is given to the wall Π1. This force causes the wall Π1 to move with a constant velocity V = V e1, V > 0, while Π2 remains fixed; for further information, see to Figure 2. There is no force from the outside in action on the liquid. The velocity and pressure fields, denoted as v = V (x2/d)e1 and p = p0=const. (pure shear flow), are able to be readily verified to meet equation (9) when f is greater than or equal to zero. Additionally, the boundary conditions v(0) = 0 and v(d) = V e1 are established.

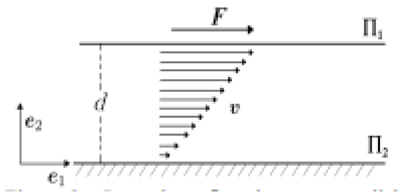


Figure 2. Pure shear flow between parallel plates induced by a force F .

Based on equations (2) and (4), we can determine that the force t per unit area that the fluid exerts on Π1 may be expressed as follows:

$$\boldsymbol{t}(x_1, d, x_3) = -\boldsymbol{e}_2 \cdot \boldsymbol{T}(x_1, d, x_3) = p_0 \boldsymbol{e}_2 - \mu \frac{V}{d} \boldsymbol{e}_1,$$

that is, t has a \purely pressure" component tp = p0 e2, and a \purely shear viscous" component tv = −μ(V /d)e1. As expected in a viscous liquid, tv is directed parallel to F and, of course, if it is not zero, it should also act against F, that is, tv · e1 < 0. However, (V /d) > 0, and so we must have μ > 0. Since the physical properties of the fluid are independent of the particular flow, this simple reasoning justifies the assumption made in (5).

method

Take, for example, the differential equation that is non-linear.

$$N[u(t)] = 0$$

where, N is a nonlinear operator, u(t) is an unknown function.

Construct the so-called zero order deformation equation as:

$$(1 - nq) L[\phi(x,t;q) - u_0(t)] = F(n) q N[\phi(x,t;q)]$$

Where F(n) is an auxillary function that is not zero, n is more than or equal to one, q ϵ [0,1] is the embedding parameter, and L is an auxillary linear operator respectively. It is necessary to select the function F(n) based on the problem that has been presented. when q equals zero and q equals one-nth of n,

$$\phi(x,t,0) = u_0(x,t) \ and \ \phi(x,t,\frac{1}{n}) = u(x,t)$$

As a result, the solution may be found as q grows from 0 to 1/n $\phi(t;q)$ varies from the initial guess to $u_0(t)$ the solution Having the freedom to choose $u_0(t)$, L, h, $H(x,t,)$, one can choose them appropriately, so that the solution $\phi(x,t;q)$ of (12) exists for $q \in [0,1/n]$. Expanding $\phi(x,t;q)$ in Taylor's series, we get:

$$\phi(x,t;q) = u_0(t) + \sum_{1}^{\infty} u_m(t)(q)^m$$

4. Results and discussion

When the parameters that determine the flow are changed, the dimensionless function f is shown for each difference. Figure 1 through Figure 4 depicts the plotting of the dimensionless capacity f against the dimensionless coordinate, where A is equal to 1, 3, 5, and 10 and where R is between 0 and. According to the graphs, it can be noted that the value of f decreases at the focus area, while it displays an increase towards the dividers of the channel as the value of R decreases. Fig.5 is a representation of the function f, where A is equal to one and R is equal to zero. As the value of R decreases, it is abundantly clear that the value of f decreases. The graphs are generated for the speed profiles of)f □ when A is equal to 0.75, 0.5, and 1 and R is greater than 0. From Figure 6 to Figure 9, it can be seen that the value of f decreases as the value of R increases. A number of different values of A and 0!!R are depicted in Figure 6 to 10. The graphs demonstrate that the value of f grows as the value of R increases.

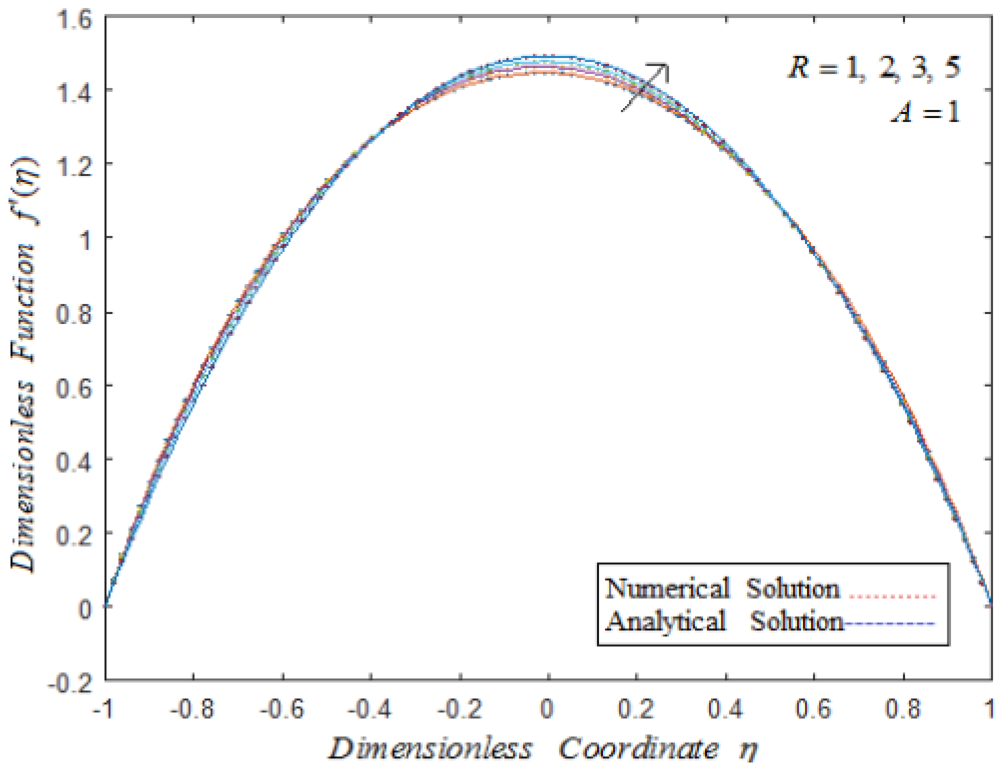


Fig.3: Dimensionless coordinate η versus velocity profile)η(′f . The curve is plotted using the eqn. (31) for fixed A=1 and varying R=1,2,3,5.

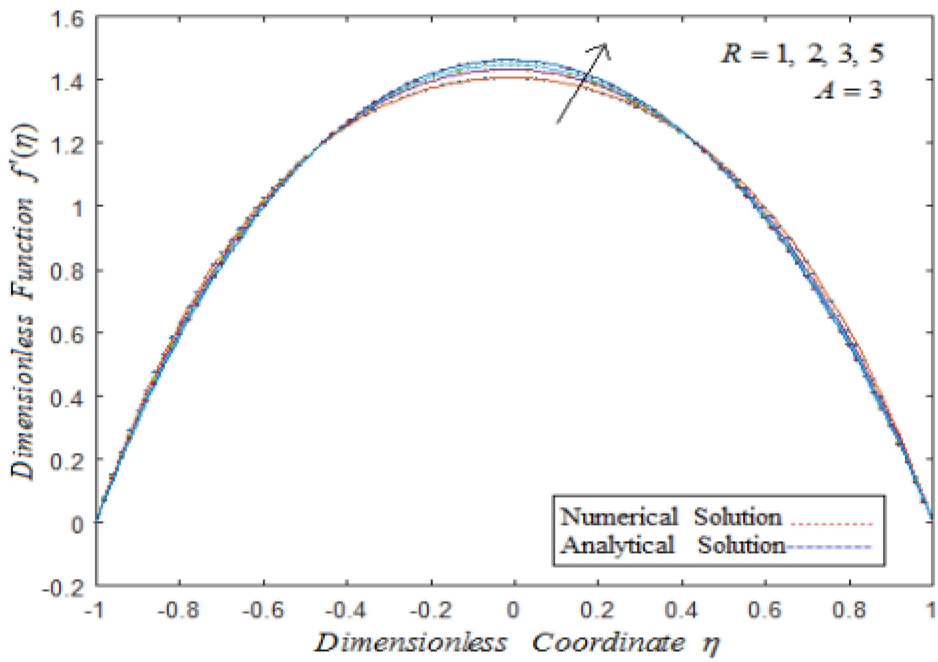


Fig.4: Dimensionless coordinate η versus velocity profile)η(′f . The curve is plotted using the eqn. (31) for fixed A=3 and varying R=1,2,3,5.

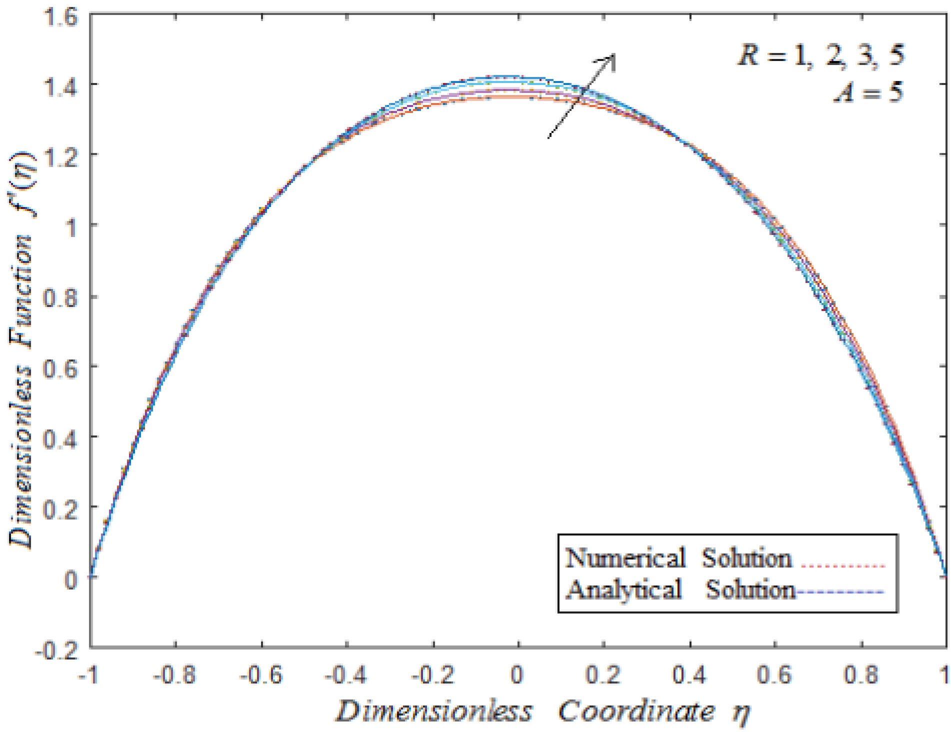


Fig.5: Dimensionless coordinate η versus velocity profile)η(′f . The curve is plotted the eqn. (31) for fixed A=5 and varying R=1,2,3,5.

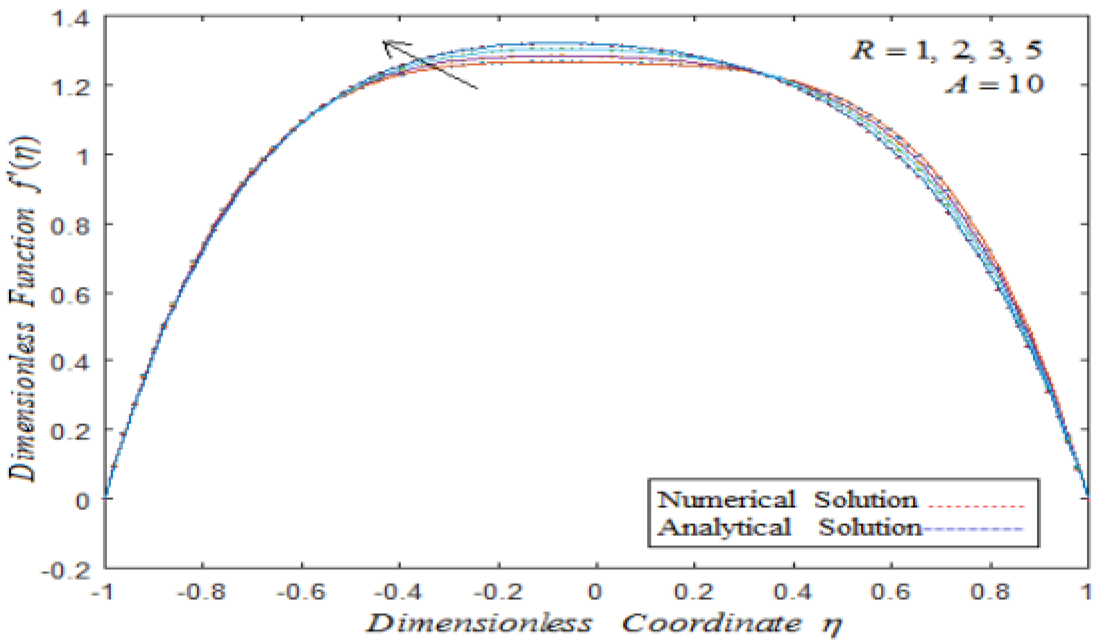


Fig.6: Dimensionless coordinate η versus velocity profile)η(′f . The curve is plotted the eqn. (31) for fixed A=10 and varying R=1,2,3,5.

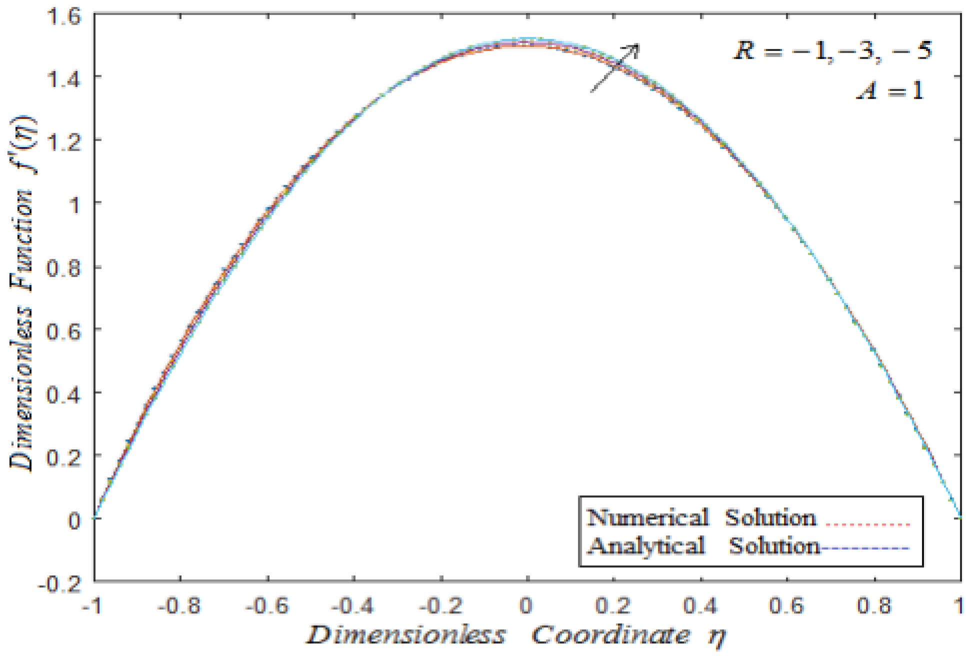


Fig.7: Dimensionless coordinate η versus velocity profile)η(′f . The curve is plotted the eqn. (31) for fixed A=1 and varying R=-1, -3, -5.

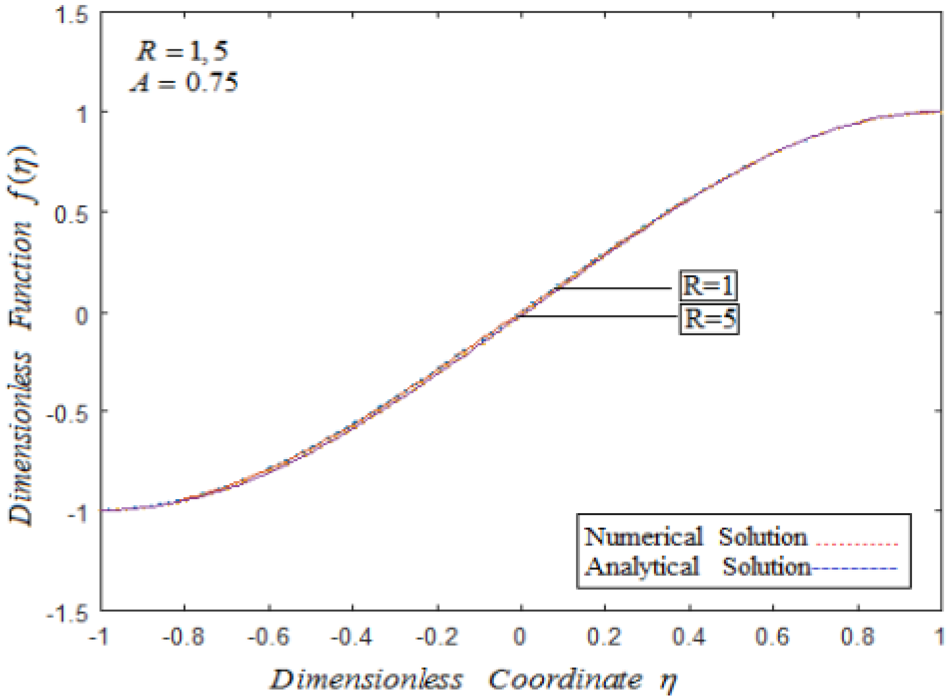


Fig.8: Dimensionless coordinate η versus Velocity profile)ηf (. The curve is plotted the eqn.(31) for fixed A=0.75 and varying R=1,5.

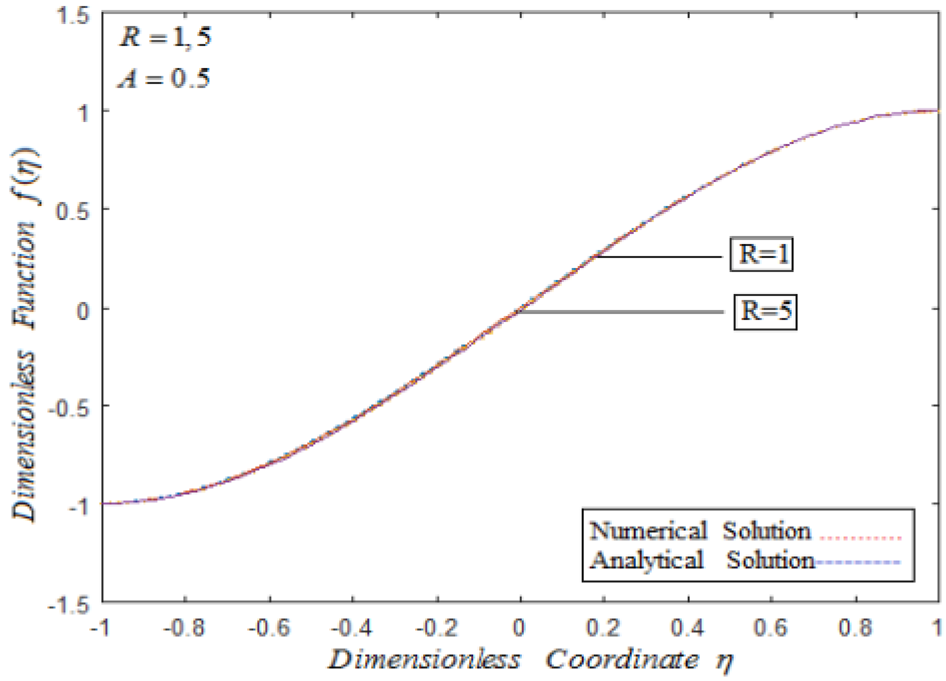


Fig.9: Dimensionless coordinate η versus Velocity profile)ηf (. The curve is plotted the eqn. (31) for fixed A=0.5 and varying R=1,5.

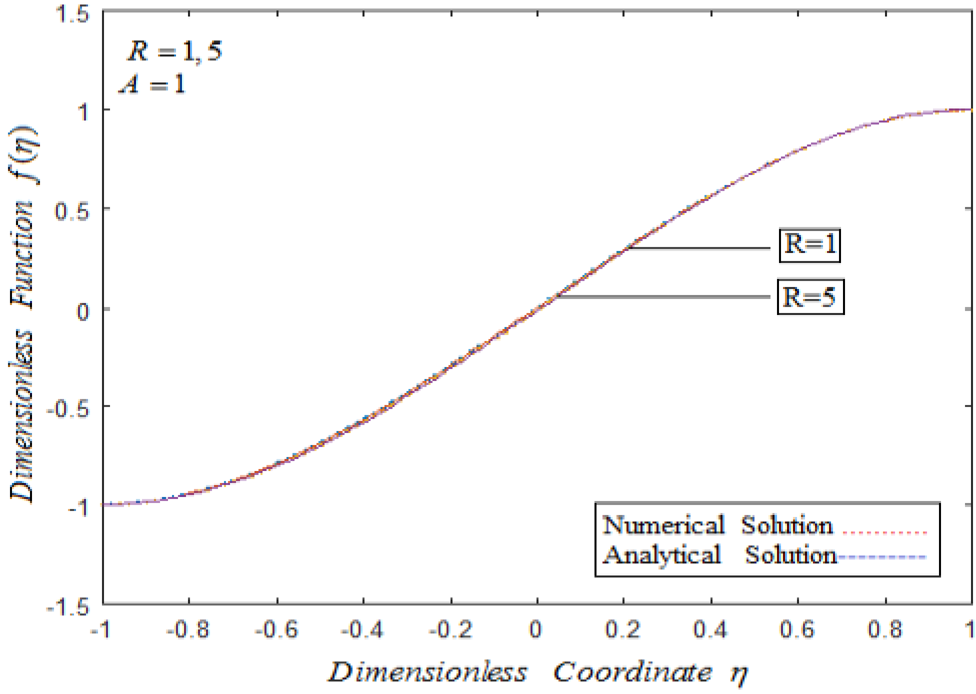


Fig.10: Dimensionless coordinate η versus Velocity profile)ηf (. The curve is plotted the eqn. (31) for fixed A=1 and varying R=1,5.

Conclusion

In addition to being essential to the comprehension of fluid behaviour, the Navier-Stokes equations have a wide range of applications in a variety of scientific and technical fields. There are substantial mathematical hurdles that are presented by these equations, despite the fact that they provide a fundamental foundation for analysing fluid motion. The presence of solutions and the regularity of such solutions, particularly in three-dimensional incompressible flows, can still be considered an open subject and continue to fascinate academics all around the world. In spite of the fact that these difficulties have not been solved, great progress has been made in approximating solutions for complicated flow problems thanks to the use of contemporary computing tools and numerical simulation procedures. The study of the Navier-Stokes equations continues to be an important endeavour, not only because of the theoretical significance of these equations but also because of the real-world applications of these equations, which incorporate anything from the development of more effective transportation systems to the comprehension of the dynamics of the climate. The discipline of fluid dynamics will continue to advance as a result of ongoing attempts to build better models of turbulence, enhance techniques for solving problems, and address the complexity of mathematical problems. In the long term, a more in-depth comprehension of these equations has the potential to improve not just our theoretical knowledge but also our capacity to tackle actual engineering challenges in a variety of different sectors.

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